

# Scalable Heisenberg-limited metrology using mixed states

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**Improving the precision of measurements is a prime challenge to the scientific community. Quantum metrology provides methods to overcome the standard quantum limit (SQL) of  $1/\sqrt{N}$  and to reach the fundamental Heisenberg-limit (HL) of  $1/N$  [1]. While a lot of theoretical [2, 3] and experimental [4, 5] works have been dedicated to this task, most of the attempts focused on utilizing NOON [6–8] and squeezed states [9–12], which exhibit unique quantum correlations. Here we present, and experimentally implement, a new scheme for precision measurements that enables reaching the HL. Our scheme is based on a probe in a mixed state with a large uncertainty, combined with a post-selection, such that the Fisher information [13] is maximized, and the Carmér-Rao bound [14, 15] is saturated. We performed a Heisenberg limited measurement of the Kerr non-linearity of a single photon, where an ultra-small Kerr phase of  $\simeq 6 \times 10^{-8}$  was observed with an unprecedented precision of  $\simeq 10^{-9}$ . Our method paves the way to Heisenberg-limited metrology with only mixed state.**

Consider that we are given a physical process that is described by an interaction Hamiltonian  $H(g)$ , which depends on a small parameter  $g$  that we want to estimate. The precision of this estimate is ultimately limited by the Carmér-Rao bound, which implies that [16]

$$\Delta g \geq \Delta g_{min} = \frac{1}{\Delta H \sqrt{\nu}}, \quad (1)$$

where  $\Delta H = \sqrt{\langle H^2 \rangle - \langle H \rangle^2}$  is the standard deviation of  $H$  and  $\nu$  is the number of times  $H$  is used. By preparing an initial state with a large  $\Delta H$ , one can improve the precision. The interaction can involve a large number,  $N$ , of subsystem. In this case  $H = \sum_{i=1}^N H_i$ , and the scaling of the precision with respect to  $N$  is of special importance. If the initial state is such that the subsystems are uncorrelated,  $\langle H_i H_j \rangle - \langle H_i \rangle \langle H_j \rangle \sim \delta_{i,j}$ , then  $\Delta H \sim \sqrt{N}$  and the precision follows the SQL, in general. However, when the subsystems are initially entangled, it is possible to have  $\Delta H \sim N$ , which yields the HL. Thus, a lot of effort was put into generating highly entangled states, such as NOON or squeezed states.

A direct inspection of Eq. (1) raises the question of whether the maximization of  $\Delta H$  to  $\Delta H \sim N$  can be achieved in a different way. In this paper we answer this question in the affirmative by introducing a novel scheme for precision measurements that is focused directly on maximizing  $\Delta H$ . This

is achieved by introducing externally induced fluctuations. In order to see how to utilize these fluctuations, we use the formalism of weak measurement [17–20]. Consider an interaction Hamiltonian  $H = f(t)PC$ , where  $P$ , which is related to a probe, and  $C$ , which is related to a system, are both Hermitian operators and  $f(t)$  is a coupling function with a finite support satisfying  $g = \int f(t)dt$ . If the system is prepared in a state  $|\psi\rangle$  before the interaction and post-selected later to a state  $|\phi\rangle$ , then  $\langle P \rangle$  will be modified by [21, 22]

$$\delta P = 2g(\Delta P)^2 \text{Im} C_w, \quad (2)$$

where  $C_w = \frac{\langle \phi | C | \psi \rangle}{\langle \phi | \psi \rangle}$  is the Weak Value of  $C$  and  $\Delta P$  is the standard deviation of  $P$ . Calculating the signal to noise ratio, where (2) is the signal and  $\Delta P$  is the noise, yields a precision  $\Delta g \sim \Delta P^{-1}$ . (The same result can be obtained by calculating the Fisher information, for details see supplementary material). As we noted before, when  $P$  pertains to  $N$  uncorrelated systems  $\Delta P \sim \sqrt{N}$  and the SQL still applies. However, it was shown [23] that (2) holds even when  $\Delta P$  is due to classical fluctuations, such as technical noise, and that this noise can improve the measurement precision by increasing  $\Delta P$ . Our method is based on this idea, and by taking the limit of the largest possible fluctuations  $\Delta P \sim \langle P \rangle$ , we obtain the HL. Our method, based on [23], is shown to be robust and we experimentally demonstrate the HL scaling and give an improved precision compared to the standard method [24].

Fig. 1 (a) illustrates, in an optical phase diagram, the standard interference method for phase estimation, using coherent states [25]. A state  $|\alpha\rangle$ , represented by a disk of diameter  $1/2$  with its center a distance of  $|\alpha|$  from the origin, acquires a phase  $g$ , and is transformed according to  $|\alpha\rangle \rightarrow |e^{ig}\alpha\rangle$ , which induces a rotation and a shift of  $\simeq g|\alpha|$  in the location of the disk. Measuring a quadrature of the state yield an estimation for  $g$ , which is limited by the uncertainty, represented by the width of the disk. Increasing the photon number  $N = |\alpha|^2$  would increase the shift and improve the precision according to the SQL  $\Delta g \sim 1/\sqrt{N}$ . The HL can be obtained with squeezed state, as shown in Fig. 1 (b). Since the area of the disk is fixed, decreasing the uncertainty in one direction imply increasing it in another. Reaching the maximum length of  $|\alpha|$ , imply a minimum width of  $1/|\alpha|$  and a precision according to the HL  $\Delta g \sim 1/N$ .

Consider now that the phase is due to an interaction with another system as shown in Fig. 1 (c). The interaction can

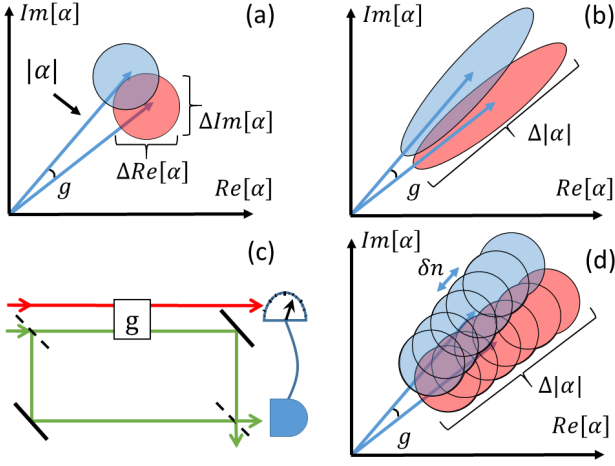


FIG. 1. **The precision to estimate coupling strength  $g$  with different meter states** (a) The standard method using coherent states  $|\alpha\rangle$ : The inherent uncertainty  $\Delta \text{Re}[\alpha] = \Delta \text{Im}[\alpha] = 1/2$  limits the precision of measuring a phase  $\text{Arg}(\alpha)$ . Increasing  $\alpha$  yields the SQL. (b) Reaching the HL with squeezed states: Reducing the uncertainty  $\Delta \text{Arg}(\alpha)$  entails an increase in  $\Delta |\alpha|$ , where the upper bound  $\Delta |\alpha| \sim |\alpha|$  is the HL. (c) The scheme: A single-photon goes through an interferometer (green), where it interacts with a strong pulse (red) in one arm. The probability for the photon to exit at some port depends on the phase it acquires by the interaction and thus also on the number of photons in the probe. Postselecting the pulses accordingly induces a shift in the average photon number, from which the interaction strength can be estimated. (d) The new method: The initial state is a statistical ensemble of  $|\alpha\rangle$ . Each member acquires a phase, dependent on  $|\alpha|$ , and interfere (with itself) yielding the probability (3). The final ensemble obtains a shift (5), in the radial direction, which is proportional to its length squared. When the length is  $\sim |\alpha|$ , we reach the HL.

be described by  $U = e^{-ig\hat{n}C}$ , where  $\hat{n}$  is the photon number operator on the probe,  $C$  is operating on the additional system, and we assume that the interaction is weak,  $g \ll 1$ . If the system is prepared in a state  $|\psi\rangle$  and post-selected later to  $|\phi\rangle$ , one can approximate the impact of the interaction as  $\langle\phi|e^{-ig\hat{n}C}|\psi\rangle \simeq e^{-ig\hat{n}C_w}$ . A coherent state would transform according to  $|\alpha\rangle \rightarrow |e^{igC_w}\alpha\rangle$ , which implies a tangential shift of  $g|\alpha|\text{Re}C_w$  and a radial shift of  $g|\alpha|\text{Im}C_w$ , yielding a precision that is still limited by the SQL. In our method, we use a statistical mixture of number states, or an ensemble of coherent state, as illustrated in Fig. 1 (d). The probability of post-selection when the probe has  $n$  photons is given by

$$p_{\phi|n} = |\langle\phi|\psi\rangle|^2 (1 + 2g\text{Im}C_w n) + o(ng)^2. \quad (3)$$

States with higher photon number entail bigger probability (or smaller, if  $\text{Im}C_w < 0$ ) so the photon-number distribution is shifted. The change in the average photon number is given by  $\delta n = 2g(\Delta n)^2 \text{Im}C_w$ , where  $\Delta n$  is the standard deviation of the initial distribution. Obtaining  $\delta n$  experimentally, entail a measurement error  $\sim \Delta n$ , so the estimation  $g = \delta n / (2(\Delta n)^2 \text{Im}C_w)$  yield a precision  $\Delta g \sim 1/\Delta n$  (other contribution to the estimation error are  $O(\delta n/\Delta n) \ll 1$ ). A

coherent state  $|\alpha\rangle$  has an uncertainty of  $\Delta n = |\alpha| = \sqrt{N}$ , for which this method yields the SQL. However, one can produce a distribution with a large deviation,  $\Delta n \simeq N$ , where  $N$  is the average photon number of the distribution, and achieve the HL.

Let us apply this method in the task of measuring the Kerr non-linearity at the single photon level. A strong pulse (prob) and a single photon (system) overlap inside a fiber where the dependence of the refractive index on the intensity of light induce both a self-phase modulation (SPM), and a cross-phase modulation (XPM). The effective Hamiltonian is [26]  $H = \tilde{g}_S \hat{n}^2 + \tilde{g} C \hat{n}$ , where  $\hat{n}$  refers now to the photon number in the probe,  $C$  is the photon number in the system and  $\tilde{g}_S$  ( $\tilde{g}$ ) is the coupling due to the SPM (XPM). Integration along the fiber yields the coefficient  $g_S$  ( $g$ ) for the SPM (XPM), such that the evolution is given by  $U = e^{-ig_S \hat{n}^2 - ig C \hat{n}}$ . The XPM represents an interaction involving a single photon and thus measuring  $g$  is highly important for many applications. An experiment aimed at this task was recently performed by Matsuda *et al.* [24], using the standard approach as described above. The main limitation in their setup came from the additional noise introduced by the SPM, which is dominated when  $N g_S \sim 1$ . Using our scheme, as we show below, the SPM is insignificant since intensity is measured instead of a phase. While the limit for our method  $N g < 1$ , is similar, the improved precision,  $\Delta g \sim N^{-1}$  rather than  $\Delta g \sim N^{-1/2}$ , entail an improvement of several orders of magnitude.

We now show in detail how to measure  $g$  using our method and analyze the resulting precision. The system photon is sent into an interferometer, with one arm containing the fiber, such that its initial state is  $|\psi\rangle = (|1\rangle + |0\rangle)/\sqrt{2}$  where  $|1\rangle$  and  $|0\rangle$  are eigenstates of  $C$  with eigenvalues 0 and 1 respectively. The probe is in a coherent state  $|\alpha\rangle$ , but by modulating the power of the laser we get a distribution of  $\alpha$ , which can be written as a mixed state  $\rho_p = \sum_{\alpha} p_{\alpha} |\alpha\rangle \langle\alpha|$ , where  $p_{\alpha}$  is the probability of having a coherent state  $|\alpha\rangle$ . After the probe goes through the fiber, we measure its photon number. However, only the trials when the system photon is found in a specific exit port are taken into account; we post-select the state of the system to be  $|\phi\rangle = (|1\rangle - e^{-i\epsilon}|0\rangle)/\sqrt{2}$ , with  $\epsilon \ll 1$ , which is set by tuning the interferometer. The result of the measurement on the probe is given by

$$\langle\hat{n}\rangle = \frac{\text{Tr}[\hat{n} U \rho_p \rho_s^{\psi} U^{\dagger} \rho_s^{\phi}]}{\text{Tr}[U \rho_p \rho_s^{\psi} U^{\dagger} \rho_s^{\phi}]}, \quad (4)$$

where  $\rho_s^{\psi} = |\psi\rangle \langle\psi|$  and  $\rho_s^{\phi} = |\phi\rangle \langle\phi|$  are the pre- and post-selected states respectively.  $\rho_s^{\phi}$  is inserted in order to represent the post-selection and this is also the reason for the normalization denominator. Since  $[U, \hat{n}] = 0$ , only the diagonal element  $\rho_p^{n,n} = \langle n | \rho_p | n \rangle$  are significant, and we can replace the trace over the probe states with a sum over photon number  $\text{Tr}_{\rho_p}[\bullet] \rightarrow \sum_n \rho_p^{n,n}[\bullet]$ , while replacing  $\hat{n} \rightarrow n$  inside the summand. Note also that the SPM part cancels completely. The trace over the system can be approximated as  $\text{Tr}[U(n) \rho_s^{\psi} U^{\dagger}(n) \rho_s^{\phi}] \simeq \epsilon^2 (1 + 2n \frac{g}{\epsilon})$  for  $n \frac{g}{\epsilon} \ll 1$ . So the

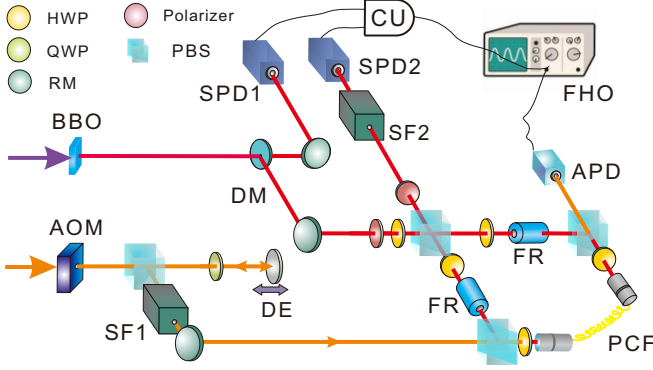


FIG. 2. **Experimental setup for measurement of the Kerr non-linearity of a single photon:** Heralded single photons (785nm) interacts with strong pulses (800nm) in the photonic crystal fiber (PCF). Postselecting the state of the single photon and measuring the photon number in the strong pulse, using full HD oscilloscope (FHO), yield an estimation for the interaction strength. For more details see the main text and supplementary material. BBO -  $\beta$ -barium borate crystal, DM - dichroic mirror, SPD - single photon detector, HWP - half wave plate, RM - reflecting mirror, FR - Faraday rotator, AOM - acoustic optical modulator, SF - spectral filter, PBS - polarized beam splitter, APD - amplified photon-diode.

change in the average photon number is given by

$$\delta n \simeq \frac{\sum_n \rho_p^{n,n} n(1 + 2n \frac{g}{\epsilon})}{\sum_n \rho_p^{n,n} (1 + 2n \frac{g}{\epsilon})} - N \simeq 2 \frac{g}{\epsilon} (\Delta n)^2. \quad (5)$$

Since  $C_w \simeq \frac{i}{\epsilon}$  Eq. (5) agrees with Eq. (2) and in the case that  $\Delta n \sim N$  we obtain the HL.

Our method, and Weak Measurements in general, require a post-selection, and for a large Weak Value the post-selection is rare. This can diminish the precision due to a decrease in  $v \sim |\langle \phi | \psi \rangle|^2$  [27, 28]. On the other hand, calculating the FI directly from (5) one obtain another amplification factor of  $\epsilon^{-2} \sim |\langle \phi | \psi \rangle|^{-2}$ , so when using the CRB (1) the dependence on  $|\langle \phi | \psi \rangle|$  cancels, in general. The precise optimal value of  $|\langle \phi | \psi \rangle|$  can be set by different technical details in the measuring equipment and the overall precision might obtain a factor of the order of unity. However, in any case, this does not affect the scaling  $\Delta g \sim N^{-1}$  which is the decisive property for the HL.

The experimental setup is shown in Fig. 2. A photon pair is generated by spontaneous down conversion. One photon is used for heralding and the other enters a polarization Sagnac interferometer (PSI), which contains a photonic crystal fiber (PCF) [29]. The single photon's polarization is prepared to be  $(|H\rangle + |V\rangle)/\sqrt{2}$  ( $H, V$  represents horizontal and vertical polarization). After entering the PSI, the photon is in an equal superposition of propagating clockwise and counter-clockwise. Only the counter-clockwise component can interact with the probe pulse, hence the system becomes  $|\psi\rangle = (|1\rangle_V + |0\rangle_H)/\sqrt{2}$ , where  $\{0, 1\}$  represents the interacting photon number. After the PSI the system is post-selected using its polarization. Faraday units cause the two compo-

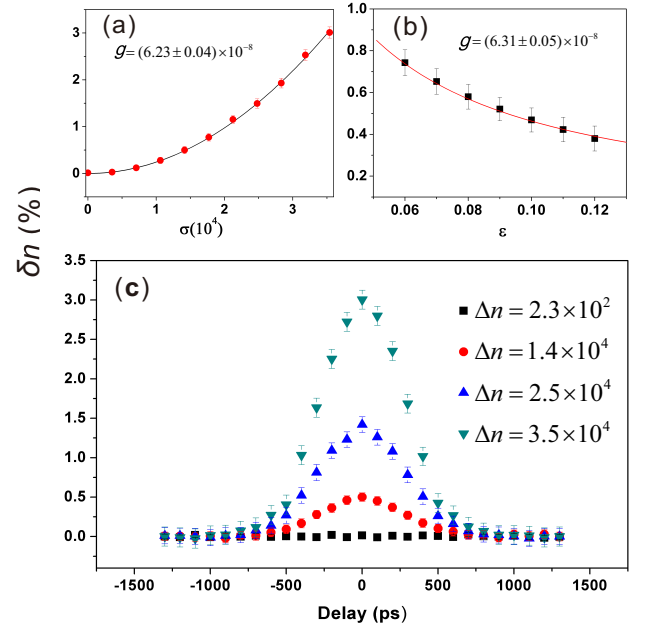


FIG. 3. **Demonstration of the validity of Eq. (5).** All plots show the relative change in the average photon number  $\delta \bar{n}$  with  $N = 5 \times 10^4$ . (a) The variance  $\Delta n^2$  is changed by controlling the magnitude of the modulation in the intensity of the strong pulse. (b) By tuning the interferometer  $\epsilon$  is changed, while  $\Delta n = 1.4 \times 10^4$ . The results in both (a) and (b) are fitted to Eq. (5) and an estimate for  $g$ , with an error due to the fitting quality, is shown in each panel. (c) A delay in the probe controls the temporal overlap with the system inside the PCF and thus tunes the interaction strength. The result trace out a Gaussian shape, corresponding to the temporal profiles of the probe and system. When the overlap is negligible  $\delta n = 0$ . This is done for several magnitudes of modulation, where in the absence of modulation the variance due to the shot noise  $\Delta n = \sqrt{N} \simeq 10^2$  is too small to observe the effect of the interaction.

nents to have the same polarization inside the PCF. Preparation of the probe, which is a strong pulse, involves (i) modulating its intensity, by an acoustic optical modulator (AOM), (ii) delaying by a translatable mirror and (iii) filtering the spectrum as not to overlap with the spectrum of the single photon. The probe then enters the PCF through a polarized beam splitter (PBS), where it overlaps with one component of the single photon and the interaction takes place. On exiting the PCF, through another PBS, the intensity of probe is measured, conditioned on the detection of both the heralding photon and the post-selected photon from the PSI. Separating the single photon from the strong pulse, after they interact, is done using both the polarization and spectral degrees of freedom. (see Methods for more details).

We start by demonstrating the validity of Eq. (5), in our system, by modifying the quantities on the RHS: the interaction strength  $g$ , the variance  $\Delta n^2$  and the Weak Value  $i/\epsilon$ , and measuring  $\delta n$ . Tuning  $g$  is done by varying the temporal overlap between the probe and the single photon. The variance is controlled by changing the magnitude of modulation in the intensity of the probe.  $\epsilon$  is set by choosing the post-selected po-

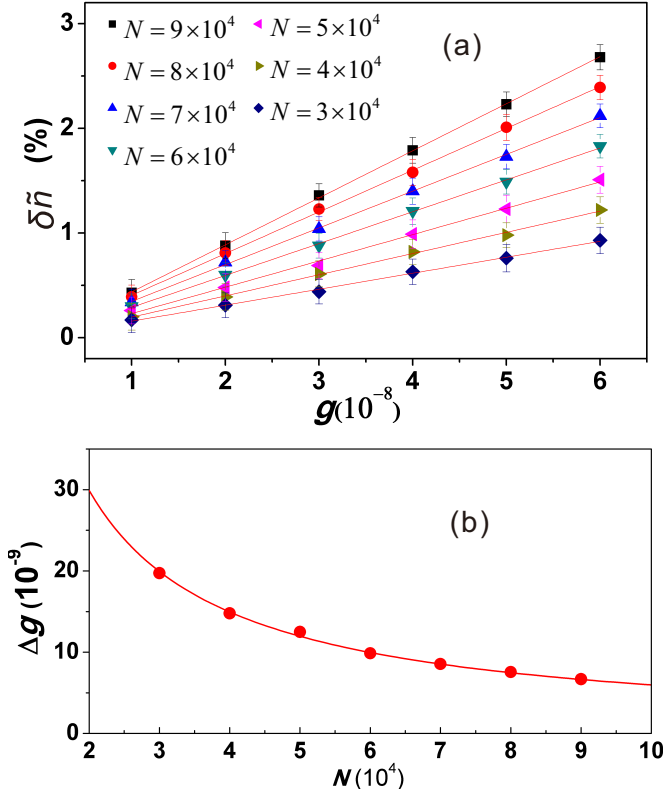


FIG. 4. **Heisenberg limited precision.** (a) The interaction strength  $g$  is changed by tuning the temporal overlap, and  $\delta\tilde{n}$  is measured for a number of values of  $N$ . The measurement precision is  $\Delta g = \Delta\delta\tilde{n} / \frac{\partial\delta\tilde{n}}{\partial g}$ , where  $\frac{\partial\delta\tilde{n}}{\partial g}$  is the slope and  $\Delta\delta\tilde{n}$  the uncertainty in  $\delta\tilde{n}$ , shown as error bars. (b) The experimental values of  $\Delta g$  is plotted as dots for each value of  $N$  and a fitting to  $N^{-1}$  is shown as solid line. The  $N^{-1}$  behaviour is because the slope  $\frac{\partial\delta\tilde{n}}{\partial g}$  increases linearly with  $N$  while the uncertainty  $\Delta\delta\tilde{n}$  is roughly independent of  $N$ . The more common scenario where  $\Delta\delta\tilde{n} \sim N^{-1/2}$  while the slope  $\frac{\partial\delta\tilde{n}}{\partial g}$  remain the same, would yield the SQL. For the relevant values of  $N$  this would be larger by two orders of magnitude.

larization state of the single photon exiting the interferometer. In Fig. 3, we plot the normalized change in photon number  $\delta\tilde{n} = \delta n / N$  in a number of ways. The results demonstrate the ability to detect the interaction of the probe with a single photon. Moreover, we show that Eq. (5) provides us with two efficient methods to estimate the interaction strength.

We turn now to demonstrating experimentally the precision of our method and in particular how the precision scales with the average photon number. Theoretically, the precision can be obtained from Eq. (5), for example by calculating the FI (see SI for details). Nonetheless, it is vital to show that the precision can be reached in practice, for large values of photon number, and that the method is indeed advantageous compared to the alternatives. In order to quantify the precision, we study the change in the measured quantity when the estimated parameter is changed. Taking into account the uncertainty of the measurement, we obtain the precision. The results, shown

in Fig. 4, demonstrate the HL, up to values of  $N \simeq 10^5$ . The maximal precision of  $\Delta g \simeq 7 \cdot 10^{-9}$  beyond a recent result for the same task [24].

The method we presented requires  $Ng \ll 1$  which implies it cannot be a single-shot measurement. The information regarding  $g$  can only be gathered from an ensemble large enough for the statistical distribution of the initial probe state to be meaningful. The method is most beneficial in the regime where  $g$  is extremely small so  $N$  can be increased immensely, while keeping the ensemble size roughly constant.

In summary, we have shown theoretically and experimentally, that mixed states, and in particular fluctuations in a probe state, can improve the precision up to the HL. Enhancing the precision with weak measurement is investigated theoretically in the context of metrology [30, 31], and some other works questioned this advantage considering the discarded resources [27, 32]. In this work, the mixed states increase the variance of the Hamiltonian, and weak measurements enable the increased variance to improve the precision. This new technique further develops the theoretical frame of weak measurement. The maximal possible magnitude of fluctuations is limited by the experimental resources, in our case the photon number. The precision scales inversely to the magnitude of fluctuations so when the resource is scaled up the precision follows the HL. Surpassing the SQL without highly entangled states indicates that it is not a fundamental limit, but an artifact of the statistics which typically describe quantum systems. The fact that our scheme is based on the utilization of mixed states allows for practical scalability. Hence, our work paves a new route to precision measurements, which can significantly modify the vast efforts devoted to this task.

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